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Statistical origin of pseudo-Hermitian supersymmetry and pseudo-Hermitian fermions

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Abstract

We show that the metric operator for a pseudo-supersymmetric Hamiltonian that has at least one negative real eigenvalue is necessarily indefinite. We introduce pseudo-Hermitian fermion (phermion) and abnormal phermion algebras and provide a pair of basic realizations of the algebra of $N = 2$ pseudo-supersymmetric quantum mechanics in which pseudo-supersymmetry is identified with either a boson-phermion or a boson-abnormal-phermion exchange symmetry. We further establish the physical equivalence (non-equivalence) of phermions (abnormal phermions) with ordinary fermions, describe the underlying Lie algebras and study multi-particle systems of abnormal phermions. The latter provides a certain bosonization of multi-fermion systems.

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1. Introduction

Supersymmetry entered theoretical physics as the symmetry allowing for the exchange or mixing of the fermionic and bosonic degrees of freedom in certain quantum field theories [1]. The subsequent attempts [2] to understand the issue of the spontaneous supersymmetry-breaking led to the discovery of supersymmetric quantum mechanics (SQM) [3, 4]. Since its inception in the early 1980s, SQM has become a focus of attention for its various physical applications [4, 5, 6] and mathematical implications [7].

The impact of SQM in theoretical physics and mathematics motivated the introduction and investigation of various generalizations of SQM. Most of these generalizations, including parasupersymmetric [8], orthosupersymmetric [9, 10], q -deformed [11] and fractional [12] supersymmetric quantum mechanics, are algebraic in nature in the sense that they are defined in terms of an underlying operator algebra that generalizes that of SQM. The main guiding principle in generalizing the algebra of SQM has been to replace the role played

by the fermionic degree of freedom in SQM with that of a parafermionic, orthofermionic or q -fermionic degree of freedom¹.

In addition to these algebraic generalizations, there is also a class of topological generalizations of supersymmetry called topological symmetries [16–19]. The latter are defined in terms of certain conditions on the spectral degeneracy structure of the Hamiltonian so that the corresponding system mimics the topological properties of the supersymmetric systems. Specifically, to each quantum system possessing a topological symmetry, there is associated a set of topological invariants that generalize the Witten index of SQM. Topological symmetries also have underlying operator algebras, and as is shown in [17, 18], the operator algebras of most of the above-mentioned algebraic generalizations of SQM may be recovered as those of the topological symmetries.

Recently, the present author has introduced yet a third class of generalizations of supersymmetry called pseudo-Hermitian supersymmetry or pseudo-supersymmetry [20, 21]. The latter may be viewed as a geometric generalization of supersymmetry, because its definition relies on allowing for the adjoint of its generator(s) to be defined in terms of a possibly indefinite inner product (hence the metric) on the Hilbert space. The main practical motivation for the introduction of pseudo-supersymmetry is that it provides a general framework [21] that encompasses all the attempts at generating non-Hermitian Hamiltonians with a real spectrum by intertwining Hermitian Hamiltonians [22].

The algebra of pseudo-SQM of order $2N$ is given by

$$\mathcal{Q}_a^2 = 0 \quad [\mathcal{Q}_a, H] = 0 \quad \{\mathcal{Q}_a, \mathcal{Q}_b^\sharp\} = 2\delta_{ab}H \quad (1)$$

where \mathcal{Q}_a with $a \in \{1, 2, \dots, N\}$ are distinct generators of pseudo-supersymmetry, H is the Hamiltonian and for any linear operator A acting in the Hilbert space (\mathcal{H}), the operator A^\sharp stands for the pseudo-adjoint of A . The latter is defined in terms of an invertible bounded Hermitian (self-adjoint) linear operator $\eta : \mathcal{H} \rightarrow \mathcal{H}$ according to [20]

$$A^\sharp := \eta^{-1} A^\dagger \eta, \quad (2)$$

where A^\dagger stands for the ordinary adjoint of A , i.e. the unique operator satisfying $\langle \cdot | A \cdot \rangle = \langle A^\dagger \cdot | \cdot \rangle$ with $\langle \cdot | \cdot \rangle$ denoting the inner product of \mathcal{H} . Clearly, for $\eta = 1$, $A^\sharp = A^\dagger$ and the algebra (1) coincides with that of SQM of order $2N$.

The operator η , which is a possibly indefinite operator², defines a possibly indefinite inner product [20, 23] on \mathcal{H} , namely $\langle \cdot, \cdot \rangle_\eta := \langle \cdot | \eta \cdot \rangle$. As a result, it is sometimes referred to as a metric operator. The pseudo-adjoint A^\sharp is the adjoint of A with respect to the inner product $\langle \cdot, \cdot \rangle_\eta$, i.e. A^\sharp is the unique operator satisfying $\langle \cdot | A \cdot \rangle_\eta = \langle A^\sharp \cdot | \cdot \rangle_\eta$.

The main purpose of this paper is to seek for the statistical origin of pseudo-supersymmetry. In particular, we will consider a Hamiltonian of the form³

$$H = E(N + \mathcal{N}) \quad (3)$$

where E is a real constant, N is the boson number operator and \mathcal{N} is the number operator for a single degree of freedom with an initially unknown algebra \mathcal{A} of creation and annihilation operators. We will then enforce the condition that this system possesses an $N = 2$ pseudo-supersymmetry. This leads to two different sets of defining relations for \mathcal{A} depending on whether the metric operator is definite or indefinite.

¹ The exchange symmetry of an order 2 parafermion–paraboson pair turns out to be a centrally extended $N = 4$ supersymmetry [13]. The q -boson–fermion and q -boson– q -fermion exchange symmetries have also been considered in the literature [14, 15].

² An indefinite operator is a Hermitian operator whose spectrum includes both strictly negative and strictly positive real numbers.

³ This is the simplest Hamiltonian that allows for the study of the statistical origin of supersymmetry and its various generalizations, in particular pseudo-supersymmetry. It is also motivated by field theoretical considerations.

The first and probably the most natural candidate for \mathcal{A} is the pseudo-Hermitian generalization of the fermion algebra, namely

$$\alpha_+^2 = \alpha_+^{\sharp 2} = 0 \quad \{\alpha_+, \alpha_+^\sharp\} = 1 \tag{4}$$

where $\alpha_+^\sharp := \eta^{-1} \alpha_+^\dagger \eta$ and α_+ are respectively the creation and annihilation operators of what we call a *pseudo-Hermitian fermion* or simply a *phermion*⁴. The corresponding number operator is given by

$$\mathcal{N}_+ = \alpha_+^\sharp \alpha_+ \tag{5}$$

so we set $\mathcal{N} = \mathcal{N}_+$.

Again for $\eta = 1$, (4) reduces to the fermion algebra. In general, as we will see, the algebra (4) does not support an indefinite metric operator η . Furthermore, for any positive- or negative-definite η , we can map (4) onto the fermion algebra by a similarity transformation. Hence in this case \mathcal{N} is equivalent to the fermion number operator and the pseudo-supersymmetry of H is equivalent to an ordinary $N = 2$ supersymmetry.

For an indefinite metric operator a simple and natural candidate for the algebra \mathcal{A} —that is compatible with the pseudo-supersymmetry of H —is the defining operator algebra of what we propose to call an *abnormal pseudo-Hermitian fermion* or simply an *abnormal phermion*. It is given by

$$\alpha_-^2 = \alpha_-^{\sharp 2} = 0 \quad \{\alpha_-, \alpha_-^\sharp\} = -1 \tag{6}$$

where α_-^\sharp and α_- are respectively the abnormal phermion creation and annihilation operators.

The above choice of the terminology has been partly adopted from a paper of Sudarshan, namely [26], on indefinite-metric quantum mechanics [27, 28], where he considers the following algebra of creation (a_-^\dagger) and annihilation (a_-) operators and refers to it as the ‘abnormal commutation relations’:

$$[a_-, a_-^\dagger] = -1. \tag{7}$$

Note that abnormal fermions—whose defining algebra would correspond to replacing \sharp by \dagger (setting $\eta = 1$) in (6)—do not exist. This is because $\{\alpha_-, \alpha_-^\dagger\}$, being a positive operator, cannot be equated to -1 . It is the notion of the pseudo-adjoint [20] that allows for considering a fermionic analogue of Sudarshan’s abnormal bosonic degrees of freedom [26].

The present paper is organized as follows. In section 2, we present a general discussion of the algebras of creation and annihilation operators and review the basic realization of $N = 2$ SQM using the Hamiltonian (3) with \mathcal{N} being the fermion number operator. In section 3, we summarize the main properties of $N = 2$ pseudo-supersymmetric systems and establish a previously unnoticed spectral consequence of pseudo-supersymmetry. In section 4, we describe the pseudo-supersymmetry of the Hamiltonian (3) for the case that \mathcal{N} is identified with the phermion number operator and show that in this case η cannot be indefinite. Here we also demonstrate the physical equivalence of phermions and fermions. In section 5, we explore the basic properties of abnormal phermions and discuss their role in obtaining a fundamental realization of the $N = 2$ pseudo-supersymmetry with an indefinite metric operator. In section 6, we elucidate the group theoretical basis of the phermion and abnormal phermion algebras. Finally in section 7, we offer a summary of our findings and present our concluding remarks.

⁴ Note that pseudo-Hermitian fermions are different from the pseudo-fermions of [24]. Similarly, there is no direct relationship between the notion of pseudo-Hermitian supersymmetry that we abbreviate as pseudo-supersymmetry [20, 21] and the boson-pseudo-fermion exchange symmetry that is also called pseudo-supersymmetry [24, 25].

2. Algebras of creation and annihilation operators and the basic realization of $N = 2$ SQM

Consider a complex $*$ -algebra⁵ generated by three elements: c, c^*, n and subject to the relations

$$n^* = n \quad [c, n] = c. \quad (8)$$

Then c^*, c and n are respectively called the creation, annihilation and number operators of a particle whose statistical properties are determined by supplementing (8) with one or more additional relations among the generators, namely

$$P_\ell(c, c^*, n) = 0 \quad \ell \in \{1, 2, \dots, r\} \quad (9)$$

where $r \in \mathbb{Z}^+$ and $P_\ell : \mathbb{C}^3 \rightarrow \mathbb{C}$ is a polynomial for each ℓ . Note that the relations obtained by applying $*$ to both sides of (8) and (9) are understood to hold as well. For example, in this way we obtain from (8) the relation

$$[c^*, n] = -c^*. \quad (10)$$

Let \mathcal{C} denote the complex $*$ -algebra generated by c, c^*, n and subject to relations (8) and (9). Suppose that \mathcal{C} admits faithful irreducible Hilbert space representations (\mathcal{H}, ρ) with Hilbert (representation) space \mathcal{H} and the representation map $\rho : \mathcal{C} \rightarrow \text{End}(\mathcal{H})$, where $\text{End}(\mathcal{H})$ stands for the complex associative algebra of linear operators acting in \mathcal{H} . Furthermore, suppose that there exists such a representation of \mathcal{C} that is also a $*$ -representation [29] with respect to some possibly indefinite [23] inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on \mathcal{H} that may differ from its defining inner product $\langle \cdot | \cdot \rangle$. This means that for all $v \in \mathcal{C}$, $\rho(v^*) = \rho(v)^\ddagger$, where \ddagger denotes the adjoint with respect to the inner product $\langle\langle \cdot, \cdot \rangle\rangle$, i.e. for all $L \in \text{End}(\mathcal{H})$, $L^\ddagger \in \text{End}(\mathcal{H})$ is the unique linear operator defined by $\langle\langle \cdot, L \cdot \rangle\rangle = \langle\langle L^\ddagger \cdot, \cdot \rangle\rangle$. Then \mathcal{C} together with (\mathcal{H}, ρ) describes a physical particle⁶, and \mathcal{C} is called the abstract algebra of creation and annihilation operators of this particle.

The typical examples of the above notion of a physical particle are bosons and fermions. The abstract operator algebra \mathcal{B} for a boson is determined by (8) and the relations

$$n = c^*c \quad (11)$$

$$[c, c^*] = 1. \quad (12)$$

Similarly, the abstract operator algebra \mathcal{F} for a fermion is determined by (8), (11) and

$$c^2 = 0 \quad (13)$$

$$\{c, c^*\} = 1. \quad (14)$$

It is not difficult to see that the above description of a physical particle applies to parafermions and parabosons [30, 31], orthofermions [9, 10], q -deformed fermions [15], q -deformed bosons [32] and their generalizations [33].

It is well known that both \mathcal{B} and \mathcal{F} have (up to equivalence) unique (unitary) $*$ -irreducible representations⁷. The unitary irreducible representation of \mathcal{B} is (up to

⁵ A complex $*$ -algebra is a complex vector space \mathcal{C} endowed with an associative multiplication (which makes it into a complex associative algebra) and a map $*$: $\mathcal{C} \rightarrow \mathcal{C}$ with the following properties. Let $z \in \mathbb{C}$ (with complex conjugate \bar{z}) and $a, b \in \mathcal{C}$ be arbitrary and denote $*(a)$ by a^* . Then (1) $(a^*)^* = a$, so that $*$ is an involution; (2) $(za)^* = \bar{z}a^*$ and $(a+b)^* = a^* + b^*$, so that $*$ is antilinear; (3) $(ab)^* = b^*a^*$ [29].

⁶ A physical particle may or may not be a fundamental particle. The qualification 'physical' means that the ensuing mathematical structure has the potential to describe physical systems displaying effective particle-like behaviour, e.g. quasi-particles of condensed matter physics.

⁷ A unitary representation is a $*$ -representation with the inner product on the representation space \mathcal{H} identified with its defining positive-definite inner product $\langle \cdot | \cdot \rangle$.

similarity transformations) given by $\mathcal{H} = L^2(\mathbb{R})$ endowed with the usual L^2 -inner product and the representation map $\rho_b : \mathcal{B} \rightarrow \text{End}(L^2(\mathbb{R}))$ (uniquely) determined by $\rho_b(c) = (X + iP)/\sqrt{2} =: a$, where X and P are normalized (dimensionless) position and momentum operators satisfying $[X, P] = i1$. The fact that $(L^2(\mathbb{R}), \rho_b)$ is a faithful unitary representation of \mathcal{B} implies that $\rho_b(1) = 1$ and $\rho_b(c^*) = \rho_b(c)^\dagger = a^\dagger$. In particular in this representation, (12) takes the form

$$[a, a^\dagger] = 1 \tag{15}$$

and the number operator n is represented by

$$N := a^\dagger a. \tag{16}$$

The unique faithful unitary irreducible representation of the fermion algebra \mathcal{F} is two dimensional. The representation space is $\mathcal{H} = \mathbb{C}^2$ equipped with the Euclidean inner product, and the representation map $\rho_f : \mathcal{F} \rightarrow \text{End}(\mathbb{C}^2)$ is (uniquely) determined by

$$\rho_f(c) := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} =: \alpha. \tag{17}$$

Again the fact that (\mathbb{C}^2, ρ_f) is a faithful unitary representation of \mathcal{F} implies that $\rho_f(1) = 1$ and $\rho_f(c^*) = \rho_f(c)^\dagger = \alpha^\dagger$. Hence, in this representation (13) and (14) respectively take the form

$$\alpha^2 = 0 \quad \{\alpha, \alpha^\dagger\} = 1 \tag{18}$$

and the fermion number operator n becomes

$$\mathcal{N} := \alpha^\dagger \alpha. \tag{19}$$

Because of the uniqueness of the faithful unitary irreducible representations of both \mathcal{B} and \mathcal{F} , we will suppress the representation maps ρ_b and ρ_f and use a for c in the case of a boson and α for c in the case of a fermion. This allows us to identify the boson and fermion algebras with (15) and (18) respectively.

Now, consider the boson–fermion oscillator [34] whose Hamiltonian is given by (3) with $E > 0$, N denoting the boson number operator (16), and \mathcal{N} labelling the fermion number operator (19). The Hilbert space \mathcal{H} of this system is $L^2(\mathbb{R}) \otimes \mathbb{C}^2$, which is isomorphic as a Hilbert space to $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

If we postulate the relative bose statistics [31]:

$$[a, \alpha] = [a^\dagger, \alpha] = 0 \tag{20}$$

the Hamiltonian (3) of the boson–fermion oscillator commutes with

$$\tau := 1 - 2\mathcal{N}. \tag{21}$$

Furthermore, in view of (18), (16), (19) and (20), we have $\tau^2 = 1$ and $\tau^\dagger = \tau$. Hence τ is a grading operator splitting the Hilbert space into a direct sum of its eigenspaces:

$$\mathcal{H}_\pm := \{\psi \in \mathcal{H} | \tau\psi = \pm\psi\} \tag{22}$$

i.e. $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. (The elements of) \mathcal{H}_+ and \mathcal{H}_- are respectively called the bosonic and fermionic (state vectors) Hilbert spaces.

The ground state of H is represented by the unique state vector $|0, +\rangle$ eliminated by both a and α . In position representation it takes the form $\pi^{-1/4} e^{-x^2/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The vectors $|n, \epsilon\rangle := (n!)^{-1/2} a^{\dagger n} (\alpha^\dagger)^{(1-\epsilon)/2} |0, +\rangle$, with $n \in \mathbb{N} := \{0, 1, 2, \dots\}$, form an orthonormal basis of \mathcal{H}_ϵ where $\epsilon \in \{-1, 1\}$. The spectrum of H consists of a nondegenerate zero eigenvalue with eigenvector $|0, +\rangle$ and doubly degenerate positive eigenvalues, $E_n = En$, with the pair of linearly independent eigenvectors $(|n, +\rangle, |n - 1, -\rangle)$, where $n \in \mathbb{Z}^+$.

It is well known that this system has an $N = 2$ supersymmetry generated by

$$\mathcal{Q} = \sqrt{2E}a^\dagger\alpha. \quad (23)$$

Using (15), (18) and (20), we can easily check that

$$\mathcal{Q}^2 = 0 \quad [\mathcal{Q}, H] = 0 \quad \{\mathcal{Q}, \mathcal{Q}^\dagger\} = 2H. \quad (24)$$

Note that here as well as in the rest of this paper we use the same symbol ‘ \dagger ’ to denote the adjoint of operators acting in $L^2(\mathbb{R})$, \mathbb{C}^2 and $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$.

As indicated by expression (23), the physical meaning of the above-mentioned supersymmetry of the boson–fermion oscillator is the symmetry allowing for the exchange of bosonic and fermionic states. This is conveniently summarized by the identity $\{\mathcal{Q}, \tau\} = 0$.

3. Consequences of pseudo-supersymmetry

The defining ingredients of $N = 2$ pseudo-SQM are the associated operator algebra:

$$\mathcal{Q}^2 = 0 \quad [\mathcal{Q}, H] = 0 \quad \{\mathcal{Q}, \mathcal{Q}^\sharp\} = 2H \quad (25)$$

and the \mathbb{Z}_2 -graded structure of the Hilbert space \mathcal{H} . The latter is specified through the existence of a grading operator $\tau : \mathcal{H} \rightarrow \mathcal{H}$ that generalizes (21) in the sense that it satisfies $\tau^{-1} = \tau = \tau^\dagger$ and hence leads to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ of the Hilbert space where \mathcal{H}_\pm are defined according to (22). Furthermore, τ anticommutes with the pseudo-supersymmetry generator \mathcal{Q} and commutes with the metric operator η :

$$\{\tau, \mathcal{Q}\} = 0 = [\tau, \eta]. \quad (26)$$

These properties of the grading operator τ allow for a canonical representation of the $N = 2$ pseudo-supersymmetry in which the restriction of \mathcal{Q} to \mathcal{H}_- vanishes. This in turn implies that \mathcal{Q}^\sharp has vanishing restriction onto \mathcal{H}_+ . The situation is best described using the following two-component representation of the Hilbert space in which a state vector $|\psi\rangle \in \mathcal{H}$ having $|\psi, \pm\rangle$ definite grading components, i.e. $|\psi\rangle = |\psi, +\rangle + |\psi, -\rangle$ with $|\psi, \pm\rangle \in \mathcal{H}_\pm$, is represented as $|\psi\rangle = \begin{pmatrix} |\psi, +\rangle \\ |\psi, -\rangle \end{pmatrix}$. In this representation we have

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix} \quad \eta = \begin{pmatrix} \eta_+ & 0 \\ 0 & \eta_- \end{pmatrix} \quad \mathcal{Q}^\sharp = \begin{pmatrix} 0 & D^\sharp \\ 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \quad (27)$$

where $D := \mathcal{Q}|_{\mathcal{H}_+}$, $\eta_\pm := \eta|_{\mathcal{H}_\pm}$, $D^\sharp := \mathcal{Q}^\sharp|_{\mathcal{H}_-} = \eta_+^{-1}D^\dagger\eta_-$, $H_+ := H|_{\mathcal{H}_+} = D^\sharp D/2$ and $H_- := H|_{\mathcal{H}_-} = DD^\sharp/2$.

Perhaps the most important feature of pseudo-supersymmetric systems is that similar to the ordinary supersymmetric systems, the nonzero eigenvalues are doubly degenerate [21]. However, the spectrum need not be nonnegative⁸. It may even include complex-conjugate pairs of eigenvalues [21]. The argument for the presence of double degeneracy for nonzero eigenvalues is identical with the one for supersymmetry: because H and τ commute, they may be simultaneously diagonalized; one may choose to work with the eigenvectors of the Hamiltonian that have definite grading. Now, suppose $|\psi_n, +\rangle \in \mathcal{H}_+$ is such an eigenvector with definite grading (+) and eigenvalue $E_n \neq 0$. Then $|\psi_n, +\rangle$ pairs with the eigenvector $|\psi_n, -\rangle := \mathcal{Q}|\psi_n, +\rangle \in \mathcal{H}_-$ which has the opposite grading (–) and the same eigenvalue E_n .

The following is another simple consequence of pseudo-supersymmetry that has, however, no supersymmetric analogue.

⁸ Negative energy eigenvalues also arise in some of the algebraic generalizations of supersymmetry such as parasupersymmetry [8] and polynomial (nonlinear) supersymmetry [35]. These generalizations do not seem to be directly related to pseudo-supersymmetry, for the latter does not restrict the Hamiltonian to be Hermitian.

Theorem. *Let H be a diagonalizable pseudo-supersymmetric Hamiltonian with a discrete spectrum and η be a metric operator defining the pseudo-adjoint \mathcal{Q}^\sharp of the pseudo-supersymmetry generator \mathcal{Q} . If H has a negative real eigenvalue, then η is necessarily an indefinite operator.*

Proof. Suppose H has nonzero real eigenvalues. Because it is diagonalizable and commutes with τ , there is a complete basis of eigenvectors of H with definite grading. Let $|\psi_n, +\rangle \in \mathcal{H}_+$ be such an eigenvector with a nonzero real eigenvalue E_n , then as we showed above so is $|\psi_n, -\rangle := \mathcal{Q}|\psi_n, +\rangle \in \mathcal{H}_-$. Now, compute

$$\begin{aligned} \langle\langle \psi_n, - | \psi_n, - \rangle\rangle_\eta &= \langle \psi_n, + | \mathcal{Q}^\dagger \eta \mathcal{Q} | \psi_n, + \rangle = \langle \psi_n, + | \eta \mathcal{Q}^\sharp \mathcal{Q} | \psi_n, + \rangle \\ &= 2E_n \langle \psi_n, + | \eta | \psi_n, + \rangle = 2E_n \langle\langle \psi_n, + | \psi_n, + \rangle\rangle_\eta \end{aligned} \tag{28}$$

where we have made use of (2) and (25). As shown in [20], $|\psi_n, +\rangle$ is η -orthogonal to all the eigenvectors of H having an eigenvalue different from E_n . Furthermore, $\langle\langle \psi_n, - | \psi_n, + \rangle\rangle_\eta = \langle \psi_n, - | \eta | \psi_n, + \rangle = \langle \psi_n, - | \eta_+ | \psi_n, + \rangle = 0$. This is best seen using the two-component representation (27). In particular, it implies that if $\langle\langle \psi_n, + | \psi_n, + \rangle\rangle_\eta = 0$, then $|\psi_n, +\rangle$ belongs to the kernel of η . This contradicts the fact that η is an invertible operator. Hence, $\langle\langle \psi_n, + | \psi_n, + \rangle\rangle_\eta \neq 0$. In view of this relation and equation (28), $\langle\langle \psi_n, + | \psi_n, + \rangle\rangle_\eta$ and $\langle\langle \psi_n, - | \psi_n, - \rangle\rangle_\eta$ have the same sign if $E_n > 0$. They have the opposite sign if $E_n < 0$. In particular, the presence of a negative real eigenvalue implies that η must be an indefinite operator. \square

Corollary. *Let H be as in the preceding theorem. If $\eta = \tau$, then all the nonzero real eigenvalues of H are negative.*

Proof. For $\eta = \tau$, $\langle\langle \psi_n, + | \psi_n, + \rangle\rangle_\eta > 0$ whereas $\langle\langle \psi_n, - | \psi_n, - \rangle\rangle_\eta < 0$. Hence according to (28), $E_n < 0$. \square

4. Phermions and $N = 2$ pseudo-supersymmetry with a definite metric

Consider realizing the algebra (25) of $N = 2$ pseudo-SQM using a Hamiltonian of the form (3) with N being the boson number operator (16). Then in view of the analogy with the boson–fermion oscillator and requiring that the pseudo-supersymmetry is an exchange symmetry of a boson and a particle having \mathcal{N} as its number operator, we are again led to an expression of the form (23) for the symmetry generator \mathcal{Q} , namely

$$\mathcal{Q} = \sqrt{2E} a^\dagger \alpha_+ \quad E > 0. \tag{29}$$

With this choice of \mathcal{Q} , we may fulfil relations (25) provided that we require the undetermined particle to be a pseudo-Hermitian fermion (phermion), whose defining algebra and number operator are respectively given by (4) and (5), and that we postulate the relative bose statistics, i.e. a and a^\dagger commute with any operator constructed out of α, α^\dagger and η .

We can easily obtain a Fock-space representation of the phermion algebra in the same way one obtains the Fock-space representation of the fermion algebra [36]. The representation space \mathcal{V} is the span of $|+\rangle := \alpha_+$ and $|-\rangle := \mathcal{N}_+ = \alpha_+^\sharp \alpha_+$. This is a two-dimensional subspace of the phermion algebra viewed as a four-dimensional complex vector space (with $\{1, \alpha_+, \alpha_+^\sharp, \mathcal{N}_+\}$ as a basis). Therefore, \mathcal{V} is isomorphic to \mathbb{C}^2 as a vector space. In the basis $\{| \pm \rangle\}$, the elements of \mathcal{V} are represented by column vectors, e.g.

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{30}$$

and linear operators acting in \mathcal{V} take the form of 2×2 matrices. In particular, if we denote the representation map by ρ_* , we find

$$\rho_*(\alpha_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \alpha \quad \rho_*(\alpha_+^\#) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \alpha^\dagger \quad (31)$$

where we have made use of (4) and (17).

Equations (31) show that the pfermion algebra (4) admits a faithful irreducible representation ρ_* that is identical with the basic representation ρ_f of the fermion algebra (18). This representation is a $*$ -representation provided that \mathbb{C}^2 is equipped with the standard Euclidean inner product. This happens if $\rho_*(\eta)$ is just the identity matrix I which in view of the fact that ρ_* is a faithful representation implies $\eta = 1$. But this corresponds to an ordinary fermion.

There are also other choices for the inner product on \mathbb{C}^2 that renders ρ_* a $*$ -representation. Let $\rho_*(\eta)$ be the associated metric operator. Then one can show using (31) and

$$\rho_*(\eta)\rho_*(\alpha_+^\#) = \rho_*(\eta)\rho_*(\eta^{-1}\alpha_+^\dagger\eta) = \rho_*(\alpha_+^\dagger)\rho_*(\eta)$$

that $\rho_*(\eta) = sI$, where $s \in \mathbb{R} - \{0\}$. This in particular shows that η is proportional to 1 and that in effect a pfermion is equivalent to an ordinary fermion.

The equivalence of a pfermion and a fermion may be stated in terms of the underlying abstract algebras: *The abstract pfermion algebra actually coincides with the abstract fermion algebra \mathcal{F} .* One way of seeing this is to note that any faithful representation of the pfermion algebra (4) is completely reducible to copies of the above-described basic representation ρ_* .⁹ Furthermore, the irreducible $*$ -representations of this algebra do not support an indefinite metric operator. A more explicit demonstration of the latter observation is provided by considering an arbitrary two-dimensional faithful representation ρ of (4), which is equivalent to ρ_* , and supposing that $\rho(\eta)$ is an indefinite operator, i.e. an indefinite invertible Hermitian matrix. One can then perform a similarity transform $S : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ that transforms $\rho(\eta)$ as [21, 37]

$$\rho(\eta) \rightarrow \sigma(\eta) := S^\dagger \rho(\eta) S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =: \sigma_3. \quad (32)$$

Under this transformation

$$\rho(\alpha_+) \rightarrow \sigma(\alpha_+) := S^{-1} \rho(\alpha_+) S = \begin{pmatrix} s & u \\ v & t \end{pmatrix} \quad (33)$$

for some $s, t, u, v \in \mathbb{C}$. Now, in view of the identities $\rho(\alpha_+)^2 = 0 \neq \rho(\alpha_+)$, we have $\sigma(\alpha_+)^2 = 0 \neq \sigma(\alpha_+)$. The latter relations fix s and t according to

$$s = -t = \pm\sqrt{uv}. \quad (34)$$

Substituting (34) in (33) and making use of (32), we have

$$\begin{aligned} \rho(\{\alpha_+, \alpha_+^\#\}) &= \{\rho(\alpha_+), \rho(\eta)^{-1}\rho(\alpha_+^\dagger)\rho(\eta)\} \\ &= S\{\sigma(\alpha_+), \sigma_3\sigma(\alpha_+^\dagger)\sigma_3\}S^{-1} = -(|u| - |v|)^2 I. \end{aligned} \quad (35)$$

In particular, $\rho(\{\alpha_+, \alpha_+^\#\})$ cannot be equated to the identity matrix I as required by (4). This shows that η cannot be an indefinite operator. That is the realization of $N = 2$ pseudo-SQM in terms of the boson–fermion oscillator only applies to the cases that η is a positive- or negative-definite operator.

For a negative-definite metric operator η , we can use the positive-definite metric operator $-\eta$ to define the pseudo-adjoint of the relevant operators. Hence without loss of generality,

⁹ This may be established using the approach of [10].

we will suppose that η is positive definite. In this case η has a unique positive-definite square root $\eta^{1/2}$ with inverse $\eta^{-1/2}$. It is an easy exercise to show that α_+ and $\alpha_+^\sharp = \eta^{-1}\alpha_+^\dagger\eta$ satisfy the pfermion algebra (4) if and only if $\alpha := \eta^{1/2}\alpha_+\eta^{-1/2}$ and $\alpha^\dagger = \eta^{1/2}\alpha_+^\sharp\eta^{-1/2}$ satisfy the fermion algebra (18). Therefore, the pfermion (4) and fermion (18) algebras are related by a similarity transformation, and pfermions have the same physical properties as the ordinary fermions. Similarly, the $N = 2$ pseudo-SQM is just another representation of the ordinary $N = 2$ SQM.¹⁰

5. Abnormal pfermions and $N = 2$ pseudo-supersymmetry with an indefinite metric

Because the $N = 2$ pseudo-supersymmetry algebra with an indefinite metric cannot be realized using a boson–pfermion oscillator, we seek for a modification of the pfermion algebra. A natural modification is suggested by (35). It is the algebra of an abnormal pfermion (6).

Using the notion of an abstract creation and annihilation algebra outlined in section 2, we define an abnormal pseudo-Hermitian fermion (abnormal pfermion) as the abstract complex $*$ -algebra Φ generated by three elements: c , c^* , n and subject to relations (8) and

$$n = -c^*c \quad (36)$$

$$c^2 = 0 \quad (37)$$

$$\{c, c^*\} = -1 \quad (38)$$

together with a faithful irreducible Hilbert-space representation (\mathcal{H}, ρ) of Φ such that this representation is a $*$ -representation if we endow \mathcal{H} with the (indefinite) inner product $\langle\langle \cdot, \cdot \rangle\rangle_\eta$ for some indefinite metric operator η acting in \mathcal{H} .

Viewing Φ as a complex associative algebra, i.e. disregarding its $*$ -structure, we may define

$$c_1 := -ic \quad c_2 := -ic^* \quad (39)$$

and check that c_1 , c_2 and $n = c_2c_1$ satisfy the defining relations of the fermion algebra \mathcal{F} , i.e. as complex associate algebras Φ and \mathcal{F} are isomorphic. This is sufficient to conclude that Φ has a unique two-dimensional faithful irreducible representation with representation map $\varrho_\star : \Phi \rightarrow \text{End}(\mathbb{C}^2)$ given by

$$\varrho_\star(c) = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} = i\alpha =: \alpha_- \quad \varrho_\star(c^*) = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} = i\alpha^\dagger = -\alpha_-^\dagger. \quad (40)$$

Clearly, this is not a $*$ -representation if we identify the inner product on \mathbb{C}^2 with the Euclidean inner product. But it is a $*$ -representation if we endow \mathbb{C}^2 with the indefinite inner product $\langle\langle \cdot, \cdot \rangle\rangle_{\sigma_3}$, where σ_3 is the diagonal Pauli matrix (32). Using this inner product to define the pseudo-adjoint \sharp , we have $\varrho_\star(c)^\sharp = \sigma_3\varrho_\star(c)^\dagger\sigma_3 = \varrho_\star(c^*)$ or simply

$$\alpha_-^\sharp = \varrho_\star(c^*). \quad (41)$$

With α_- and α_-^\sharp given by (40) and (41) and recalling that they provide the unique faithful irreducible $*$ -representation of Φ , we will respectively identify c and c^* with α_- and α_-^\sharp , speak of (6) as the abnormal pfermion algebra and let

$$\mathcal{N}_- := -\alpha_-^\sharp\alpha_- \quad (42)$$

be the abnormal pfermion number operator.

¹⁰ This result is consistent with those of [38, 39] where the pseudo-Hermitian Hamiltonians with a definite metric operator (that is the so-called quasi-Hermitian Hamiltonians [28, 37]) are related to Hermitian Hamiltonians via similarity transformations.

Now, we are in a position to explore the Hamiltonian (3) with N and \mathcal{N} respectively identified with the boson number operator (16) and the abnormal phermion number operator (42). Postulating the relative bose statistics, i.e. that a and a^\dagger commute with any operator constructed out of α_- , α_-^\dagger and $\eta = \sigma_3$, we can easily check that H together with

$$\mathcal{Q} =: \sqrt{2|E|}a^\dagger\alpha_- \quad \text{and} \quad E < 0 \quad (43)$$

satisfy the algebra (25) of $N = 2$ pseudo-SQM with the indefinite metric operator $\eta = \sigma_3$. Note that the grading operator for this system is again given by $\tau = 1 - 2\mathcal{N}_- = \sigma_3$. Therefore, the hypothesis ($\eta = \tau$) of the corollary given in section 3 is satisfied and the nonzero real eigenvalues of H must be negative. Indeed it is not difficult to check that the eigenvalues of H are real and non-positive. They are given by $E_n = nE = -n|E|$ where $n \in \mathbb{N}$.¹¹

The realization of the $N = 2$ pseudo-SQM in terms of a boson–abnormal-phermion exchange symmetry as outlined above enjoys a uniqueness property in the sense that considering an arbitrary two-dimensional irreducible representation of Φ with an arbitrary indefinite metric operator (2×2 indefinite invertible Hermitian matrix) on the representation space leads to an equivalent description of the abnormal phermion and the associated boson–abnormal-phermion exchange pseudo-supersymmetry. This is because, as noted in section 4, any such metric operator may be transformed to σ_3 via a similarity transformation of the Hilbert space [21, 37].

It is not difficult to see that if we adopt the basic representations of ordinary fermions ρ_\star and abnormal phermions ϱ_\star with the metric operators given by $\rho_\star(\eta) = I$ and $\varrho_\star(\eta) = \sigma_3$, then the corresponding number operators coincide: $\mathcal{N}_- = \mathcal{N}_+$. This does not however mean that a fermion and an abnormal phermion are equivalent. The distinction lies in the interpretation of the abnormal phermion state described by the state vector $|- \rangle$ of (30) that satisfies $\langle\langle -, - \rangle\rangle_{\sigma_3} = \langle\langle -|\sigma_3|- \rangle\rangle = -1$. This state vector does not belong to the physical Hilbert space, for the latter includes besides the zero vector only the state vectors with positive real norms [26–28]. Therefore, unlike a fermion that has two physical states, an abnormal phermion has a single physical state. This in turn implies that an ordinary quantum mechanical system consisting of only a single abnormal phermionic degree of freedom is trivial. Nontrivial systems may however be constructed by combining an abnormal phermion with other particles or using more than one abnormal phermion¹².

As an example, consider a system consisting of ℓ abnormal phermions with annihilation, creation, number and metric operators $\alpha_-^{(i)}$, $\alpha_-^{(i)\dagger}$, $\mathcal{N}_-^{(i)} := -\alpha_-^{(i)\dagger}\alpha_-^{(i)}$ and $\eta^{(i)}$, respectively. Suppose that for all $i \in \{1, 2, \dots, \ell\}$, $\eta^{(i)} = \sigma_3$. Clearly, the Hilbert space of this system is 2^ℓ dimensional; a set of basis vectors is given by

$$|v_1, v_2, \dots, v_\ell\rangle := (\alpha_-^{(1)\dagger})^{v_1} (\alpha_-^{(2)\dagger})^{v_2} \dots (\alpha_-^{(\ell)\dagger})^{v_\ell} |0, 0, \dots, 0\rangle \quad (44)$$

where $v_i \in \{0, 1\}$, $|0, 0, \dots, 0\rangle := |0\rangle^{(1)} \otimes |0\rangle^{(2)} \otimes \dots \otimes |0\rangle^{(\ell)}$, is the vacuum state vector for the system, $|0\rangle^{(i)}$ is the vacuum state vector for the i th abnormal phermion: $\alpha_-^{(i)}|0\rangle^{(i)} = 0$, and we adopt the relative fermi statistics, i.e. for all $i, j \in \{1, 2, \dots, \ell\}$

$$\{\alpha_-^{(i)}, \alpha_-^{(j)}\} = 0 \quad \{\alpha_-^{(i)}, \alpha_-^{(j)\dagger}\} = -\delta_{ij}. \quad (45)$$

¹¹ The fact that energy spectrum and therefore the Hamiltonian is bounded above but not below may be used to argue that this Hamiltonian does not describe a physical system. The problem with arbitrarily large negative energies may be avoided using Feynman's idea of associating this Hamiltonian with a system that evolves backward in time. This is equivalent to considering $-H$ as the Hamiltonian for the corresponding forward evolution in time. Although $-H$ coincides with the boson–fermion Hamiltonian, it describes a fundamentally different system as we explain below.

¹² Alternatively, one may associate the unphysical state with the physical state of another particle, e.g. the corresponding anti-particle.

The η -inner product of two basis vectors (44) is defined according to

$$\begin{aligned} \langle\langle \mu_1, \mu_2, \dots, \mu_\ell | \nu_1, \nu_2, \dots, \nu_\ell \rangle\rangle_\eta &:= \langle \mu_1 | \eta^{(1)} | \nu_1 \rangle \langle \mu_2 | \eta^{(2)} | \nu_2 \rangle \cdots \langle \mu_\ell | \eta^{(\ell)} | \nu_\ell \rangle \\ &= (-1)^{\nu_1 + \nu_2 + \cdots + \nu_\ell} \delta_{\mu_1, \nu_1} \delta_{\mu_2, \nu_2} \cdots \delta_{\mu_\ell, \nu_\ell} \end{aligned} \quad (46)$$

where $|\nu_i\rangle := (\alpha_-^{(1)\sharp})^{\nu_i} |0\rangle^{(i)}$. In particular, the state vectors associated with an even number of particles have a positive real η -norm. These span the physical Hilbert space $\mathcal{H}_{\text{phys}}$ of the system which is $2^{\ell-1}$ dimensional. The physical state vectors may be constructed from the vacuum state vector $|0, 0, \dots, 0\rangle$ and the ‘physical’ creation operators: $\alpha_{ij}^+ := \alpha_-^{(j)\sharp} \alpha_-^{(i)\sharp}$ with $i < j$. These together with the ‘physical’ annihilation operators $\alpha_{ij} := \alpha_-^{(i)} \alpha_-^{(j)}$ satisfy, for all $i < j$ and $k < l$,

$$[\alpha_{ij}, \alpha_{kl}] = [\alpha_{ij}^+, \alpha_{kl}^+] = 0 \quad (47)$$

$$[\alpha_{ij}, \alpha_{kl}^+] = \delta_{ik} \delta_{jl} - \delta_{ij} \delta_{jk} + \delta_{ik} \beta_{lj} + \delta_{jl} \beta_{ki} - \delta_{jk} \beta_{li} - \delta_{il} \beta_{kj} \quad (48)$$

where we have made use of (45) and introduced

$$\beta_{ij} := \alpha_-^{(i)\sharp} \alpha_-^{(j)}.$$

Note that the shift operators β_{ij} commute with the total number operator $\mathcal{N}_{\text{tot}} = \sum_{i=1}^{\ell} \mathcal{N}_-^{(i)}$. Hence they relate different (physical) states with the same number of particles.

Clearly, the same physical Hilbert space may be obtained using a system of $\ell - 1$ fermions. But then different states are related by fermionic creation and annihilation operators that satisfy anti-commutation relation. In contrast, the above description using the abnormal phermions leads to a set of creation, annihilation and shift operators that satisfy commutation relations. As a result, it makes the underlying (Lie) group structure and the associated symmetries of the system more transparent. The use of abnormal phermions seems to provide a certain type of ‘bosonization’ of the fermionic systems¹³.

6. Group theoretical basis of phermion and abnormal phermion algebras

The algebras of the phermions (4) and abnormal phermions (6) may be expressed in the unified form

$$\alpha_\epsilon^2 = 0 \quad \{\alpha_\epsilon, \alpha_\epsilon^\sharp\} = \epsilon 1 \quad (49)$$

with $\epsilon \in \{-, +\}$. The phermion (5) and abnormal phermion (42) number operators

$$\mathcal{N}_\epsilon = \epsilon \alpha_\epsilon^\sharp \alpha_\epsilon \quad (50)$$

satisfy

$$[\alpha_\epsilon, \mathcal{N}_\epsilon] = \alpha_\epsilon \quad [\alpha_\epsilon^\sharp, \mathcal{N}_\epsilon] = -\alpha_\epsilon^\sharp. \quad (51)$$

Furthermore, in the basic two-dimensional representations ρ_\star and ϱ_\star with $\rho_\star(\eta) = 1$ for phermion (so that it is just a fermion) and $\varrho_\star(\eta) = \sigma_3$ for the abnormal phermion, we can easily check using (17), (40) and (41) that

$$[\alpha_\epsilon, \alpha_\epsilon^\sharp] = 1 - 2\epsilon \mathcal{N}_\epsilon. \quad (52)$$

¹³ A similar argument applies to a parafermionic description of the physical Hilbert space using a parafermion of order $2^{\ell-1} - 1$. But the corresponding operator algebra would involve complicated ternary relations. One may also try to obtain a realization of parafermionic operators (with the above order) in terms of the abnormal phermionic operators. This may be viewed as a ‘bosonization’ of the associated parafermionic system.

Now, let us introduce the pseudo-Hermitian operators [20]

$$J_1^\epsilon := \frac{1}{2}(\alpha_\epsilon + \alpha_\epsilon^\#) \quad J_2^\epsilon := \frac{1}{2i}(\alpha_\epsilon - \alpha_\epsilon^\#) \quad J_3^\epsilon := -\mathcal{N}_\epsilon + \frac{1}{2} \quad (53)$$

and express (51) and (52) in terms of $J_{\pm i}$. This yields, for all $i, j \in \{1, 2, 3\}$,

$$[J_i^\epsilon, J_j^\epsilon] = i \sum_{k=1}^3 \delta_k^\epsilon \epsilon_{ijk} J_k^\epsilon \quad (54)$$

where $\delta_k^+ := 1$, $\delta_k^- := (1 - 2\delta_{3,k})$, $\delta_{i,j}$ is the Kronecker delta function and ϵ_{ijk} is the totally antisymmetric Levi-Civita symbol (with $\epsilon_{123} = 1$).

Equations (54) are the defining relations for the Lie algebra $su(2) = so(3)$ for $\epsilon = +$ (i.e. for phermion/fermion) and $su(1, 1) = so(2, 1)$ for $\epsilon = -$ (i.e. for abnormal phermion). This yields a direct correspondence between the abstract fermion algebra \mathcal{F} and the Lie algebra $su(2)$ and the abstract abnormal-phermion algebra Φ and the Lie algebra $su(1, 1)$. It further suggests a potential application of abnormal phermions in the study of Klein–Gordon-type field equations [40, 41], for the effective Hamiltonian in the two-component formulation of these equations involves the elements of $su(1, 1)$ [42].

7. Conclusion

The recent study of pseudo-Hermitian operators [20, 38] has its roots in an attempt to understand the mathematical origin of the surprising spectral properties of PT -symmetric Hamiltonians [43]. It has not only provided the means for a more realistic assessment of the role of PT -symmetry (and other antilinear symmetries) [39, 44] but has found applications in various other problems [40, 41, 45]. Among the most natural outcomes of this study is the formulation of the pseudo-supersymmetric quantum mechanics [20, 21]. This is a genuine generalization of ordinary supersymmetric quantum mechanics with similar topological properties [21]. It allows for a more general class of factorizations of a given pseudo-Hermitian and in particular Hermitian Hamiltonian. In this paper we elucidated the consequences of the presence of negative real eigenvalues for a pseudo-supersymmetric Hamiltonian and explored the statistical origin of the pseudo-supersymmetric quantum mechanics.

The simplest oscillator realizations of pseudo-supersymmetry involve the phermionic and abnormal phermionic degrees of freedom depending on whether the associated metric operator is definite or indefinite. We showed that a phermion is physically equivalent to an ordinary fermion. The situation is quite different for an abnormal phermion, for only half of the states of abnormal phermionic systems correspond to physical states. The latter correspond to states with an even number of abnormal phermions. They are related via a set of composite creation and annihilation operators that together with a set of shift operators satisfy certain commutation relations. The physical sector of a system of ℓ abnormal phermions may be described by $\ell - 1$ ordinary fermions. However, the latter description involves anticommutation relations and makes the study of the underlying Lie group structure of the system obscure.

Another interesting outcome of the present study is related to the association of the compact Lie algebra $su(2) = so(3)$ with ordinary fermions and the noncompact Lie algebra $su(1, 1) = so(2, 1)$ with abnormal phermions. This suggests a possible application of abnormal phermions in physical problems that have $su(1, 1)$ as a kinematical, dynamical or symmetry group [46].

The introduction of the concept of an abnormal phermion as offered in the present paper raises various related issues. We close this paper by commenting on a few of the most notable of these.

1. *Classical abnormal fermionic degrees of freedom and their quantization.* Similar to the case of abnormal bosonic degrees of freedom [26], abnormal fermions and normal fermions (fermions) both have the usual Grassmann (odd super number [34]) variables as their classical counterpart. The choice of abnormal anticommutation relations (6) over the normal anticommutation relations (18) may be viewed as an alternative way of quantizing a classical fermionic degree of freedom. In particular, one may quantize a classical supersymmetric system to obtain a quantum pseudo-supersymmetric system by employing the abnormal quantization scheme for classical fermionic degree(s) of freedom and normal quantization scheme for the bosonic degree(s) of freedom. Indeed, an interesting direction of further research would be to use the approach of [26] to construct concrete examples of systems with normal and abnormal fermionic degrees of freedom and investigate their field theoretical analogues.
2. *Abnormal fermion–abnormal boson exchange symmetry.* By conducting abnormal quantization of both fermionic and bosonic degrees of freedom of a classical supersymmetric system, one obtains a quantum system with abnormal fermionic and abnormal bosonic degrees of freedom and a quantum analogue of the classical supersymmetry transformations relating them. A natural question is to investigate the nature (operator algebra) of this symmetry.
3. *Multi abnormal fermionic versus parafermionic systems.* One can use the physical creation and annihilation operators for multi-abnormal fermionic systems to obtain a realization of those of a parafermionic system of appropriate order. This may be viewed as an alternative to Green's ansatz [47, 31] and orthofermionic [10, 18] constructions of the latter. It may also provide the means to describe the hidden supersymmetries of the corresponding parafermionic systems [48] and investigate their analogues for general multi-abnormal fermionic systems.

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Note added. After the completion of this project, [49] was brought to my attention (by one of the referees) where the authors use the abnormal bosonic degrees of freedom to formulate a bosonic analogue of the Dirac sea for fermions. The complexification scheme used in [49] is similar to the one given in equations (39).

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